Riemann's Rearrangement Theorem in the Light of Hyperreal Numbers

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Riemann's rearrangement theorem has long been used to demonstrate that conditionally convergent series do not establish a single, coherent value. The theorem states that by simply regrouping the members of a conditionally divergent series, that a person can make that series converge to essentially any real number, or to diverge. Here, we will show that the problems usually associated with the rearrangement theorem disappear when using hyperreal numbers.

1 Divergent Series Summation with Hyperreal Numbers

[1] showed that divergent series can be given definitive, distinct values by using hyperreal numbers. Hyperreal numbers are numbers which include both infinities and infinitesimals, and in which infinities and infinitesimals behave largely the same as real numbers. The hyperreal transfer principle states that every first-order statement about the reals also applies to the hyperreals.

When dealing with hyperreal numbers, since infinities and infinitesimals cannot be presented as a real number, a symbol is often chosen for a "landmark" infinity. We choose ω for this purpose. In hyperreal numbers, infinities are very distinct from each other. ω is distinct from $\omega + 1$, which is distinct from $3\omega^2 - 2\omega$. However, a hyperreal value can be simplified by finding the principal part—the term with the highest power of ω.

In the system used here, referred to as the BGN method, rather than an infinite sum going to an arbitrary infinity, it instead goes to a specific hyperreal value for infinity. The identity

$$\sum_{1}^{\omega} 1 = \omega \tag{1}$$

is used as the reference point for other summations.

Because of the transfer principle, ordinary summation rules can be applied. For instance, the sum of the series $(1 + 2 + 3 + \ldots)$ can be represented as

$$\sum_{k=1}^{\omega} k. \tag{2}$$

Using the standard summation formula for arithmetic series, this gives the series a value of

$$\sum_{k=1}^{\omega} k = \frac{\omega^2}{2} + \frac{\omega}{2}. \tag{3}$$

The principal part of this is $\frac{\omega^2}{2}$. This same rule works for convergent series. Using the standard summation rule for geometric series we can see that

$$\sum_{k=1}^{\omega} 1 \cdot \left(\frac{1}{2})\right)^k = 2 - 2^{1-\omega}. \tag{4}$$

Here, the principle part is 2, which is the same as what is indicated by the convergent series.

Using hyperreals with the BGN method unifies the theory behind convergent, divergent, and, as we will see in the present article, conditionally convergent series.

2 The Riemann Rearrangement Theorem

A conditionally convergent series is one whose sum converges, but the some of the absolute values of the series diverges. For instance, Grandi's series, which is $(1 - 1 + 1 - 1 + 1 - 1 + \ldots)$, is conditionally convergent. The series converges to $\frac{1}{2}$, but the taking the absolute value of each member of the series causes it to be divergent.

The Riemann rearrangement theorem states that such series can be rearranged to converge to *any* real value. Normally, we consider the rearrangements of members of a series to be a non-operation, but the Riemann rearrangement theorem shows that, for infinite series, this is not necessarily the case.

As an example, Grandi's series can be rearranged and regrouped to be

$$(1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots) = (1 + 0 + 0 + 0 + \ldots) = 1. \tag{5}$$

This is different from its conditionally convergent sum of $\frac{1}{2}$. Also, by regrouping, it can be made to be zero.

$$((1 - 1) + (1 - 1) + (1 - 1) + \ldots) = (0 + 0 + 0 + \ldots) = 0 \tag{6}$$

According to the Riemann Rearrangement Theorem, because subsets of this series can be continually rearranged and regrouped, we can make this converge to any value we desire.

3 The Hyperreal Answer to the Rearrangement Theorem

Ultimately, the reason the rearrangement works is because infinity (∞), as is usually expressed when not using hyperreal values, is extremely ambiguous.

When making rearrangements, one is essentially changing the number of values in the series. For instance, with Grandi's series, if it is rewritten according to the standard form given in (1), becomes

$$\sum_{k=1}^{\omega}(-1)^{k-1}. \tag{7}$$

The rearrangement given in (6) does two problematic things:

1. It changes the number of elements in the summation (from ω to $\frac{\omega}{2}$).
2. It assumes that there are an even number of values in the series, so that each 1 has a corresponding -1.

In fact, an inspection of (5) and (6) shows that each rearrangement is actually positing a *different* number of total members of the series, since (5) is assuming an odd number of values while (6) is assuming an even number of values.

In other words, rearrangements and regroupings change the values of such series, not because they intrinsically do so, but because the rearranger is playing off of the ambiguity of ∞ in the real numbers, and *changing* the number of members of the series to suit their needs. When a fixed, unambiguous infinity is attached (such as done here with ω), then the rearrangement games no longer work.

The Riemann Rearrangement Theorem relies on conditionally convergent series—series that alternate between positive and negative numbers. Without having a fixed infinite value, manipulators can alter the series by pulling as many positive or negative values out of the hat as they wish, thus causing the series to converge on whatever value they wish. This problem doesn't occur with convergent series because any values from the "infinite" portion of the series is infinitely small, and re-adding finite portions to the beginning of the series, while it will effect the exact value of the series, will not affect the principal part of the series.

However, when using hyperreal numbers, infinities are no longer ambiguous. Therefore, a series of length ω and $\omega - 1$ or $\frac{\omega}{2}$ have fundamentally different lengths, and cannot be treated as if they were the same series. Using hyperreal numbers, the Riemann Rearrangement Theorem becomes invalid, and convergent, divergent, and conditionally convergent series all exist within a single, unified understanding of series.

References

[1] J. Bartlett, L. Gaastra, and D. Nemati, "Hyperreal numbers for infinite divergent series," *Communications of the Blyth Institute*, vol. 2, no. 1, pp. 7–15, 2020.