The Interpretation of Total Differentials in Multivariable Derivatives

Jonathan Bartlett

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Many people are confused as to the meaning of a total differential in multivariable calculus. The goal of the present paper is to clear up the meaning, and show that derivatives, differentials, and the like, continue to work correctly in multivariable calculus, and can be manipulated in an algebraic manner. While partial differentials are an important topic of multivariable calculus, they are not considered here (see [1] for a discussion of them).

1 First Derivatives with Total Differentials

To begin with, let us take the differential of a simple three-variable system:

$$z = xy + y^2 - x \tag{1}$$

Taking the differential of this equation yields:

$$\mathrm{d}z = x\,\mathrm{d}y + y\,\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}x \tag{2}$$

From here, we can solve for any derivative we like, merely by moving it to one side of the equation. For instance, we can solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by merely moving all the $\mathrm{d}y$ terms by themselves, factoring out the $\mathrm{d}y$, and then dividing by the other factor and $\mathrm{d}x$:

$$\begin{aligned}
\mathrm{d}z &= x\,\mathrm{d}y + y\,\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}x \\
\mathrm{d}z - y\,\mathrm{d}x + \mathrm{d}x &= x\,\mathrm{d}y + 2y\,\mathrm{d}y \\
\mathrm{d}z - y\,\mathrm{d}x + \mathrm{d}x &= (x + 2y)\,\mathrm{d}y \\
\frac{\mathrm{d}z - y\,\mathrm{d}x + \mathrm{d}x}{(x + 2y)\,\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned} \tag{3}$$

While (3) is technically true, it is hard to interpret. However, by splitting the fraction on the left-hand side, we can get a series of derivatives:

$$\frac{1}{x + 2y}\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{y}{x + 2y} + \frac{1}{x + 2y} = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{4}$$

This looks more like calculus, but what does this equation mean? What it means is that the derivative—the ratio of changes in y to changes in x, is dependent not only on the values of x and y, but also on the ratio of changes in z to x. In other words, the slope between y and x is dependent to some degree on the choice of slope that I use to think about the slope between z and x.

To imagine this, think about tangency in three dimensions. In three dimensions, there is not a tangent *line* but a tangent *plane*. The entirety of the plane is tangent to the graph. Now imagine drawing a line on that plane that intersects the point of tangency. This is *a* tangent line, but it is not a *unique* tangent line. Imagine spinning this line around on the plane, but attached to the point of tangency. This will go through every possible slope *on the plane* in which it is contained, but not every possible slope in three dimensions. When choosing a particular slope on the plane, that will translate into a particular slope in three dimensions. There are a variety of ways of expressing this slope, but one of them is to list out the slopes between all of the variables in the three dimensions.

So, if you were to establish an x and y value for the equation above, it would give you a formula for relating the slope of y and x given a particular choice for a slope of z and x.

2 Higher Derivatives with Total Differentials

Taking higher derivatives is also possible, but you have to take into consideration an alternate notation for the second derivative. As shown in [2], the typical notation for the second derivative is not algebraically manipulable, and, instead, one must use the following notation which is derived from applying the quotient rule to the first derivative:

$$\frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} \tag{5}$$

When doing higher order differentials, the rules are the same, but one must keep in mind that the differential itself is a term. So, the differential of, say, $y\,dx$ would be found using the product rule, resulting in $y\,\mathrm{d}(\mathrm{d}x) + \mathrm{d}y\,\mathrm{d}x$. Additionally, $\mathrm{d}(\mathrm{d}x)$ is usually written instead as d^2x.

We can do this in one of two ways. Either we can go back to the first differential, (2), differentiate from there, and then solve for (5), or, we can start with the complete first derivative, (4), and take the derivative from there. We will opt for the second option, as the result is more easily interpretable.

$$\frac{dy}{dx} = \frac{1}{x+2y}\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{y}{x+2y} + \frac{1}{x+2y} \tag{6}$$

$$\frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{x+2y}\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{y}{x+2y} + \frac{1}{x+2y}\right)}{\mathrm{d}x} \tag{7}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = \frac{\mathrm{d}\left(\frac{1}{x+2y}\frac{\mathrm{d}z}{\mathrm{d}x}\right) - \mathrm{d}\left(\frac{y}{x+2y}\right) + \mathrm{d}\left(\frac{1}{x+2y}\right)}{\mathrm{d}x} \tag{8}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = \frac{\frac{1}{x+2y}\left(\frac{\mathrm{d}^2z}{\mathrm{d}x} - \frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x}\right) - \left(\frac{1}{x+2y}\right)^2(\mathrm{d}x + 2\,\mathrm{d}y)\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{x\,\mathrm{d}y + 2y\,dy - y(\mathrm{d}x + 2\,\mathrm{d}y)}{(x+2y)^2} - \frac{1}{(x+2y)^2}(\mathrm{d}x + 2\,\mathrm{d}y)}{\mathrm{d}x} \tag{9}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} = \frac{1}{x+2y}\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - \frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2}\right) - \left(\frac{1}{x+2y}\right)^2\left(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(x+2y)^2} - \frac{1}{(x+2y)^2}(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}) \tag{10}$$

$$\tag{11}$$

As is evident, taking a total second derivative of even a fairly straightforward equation gets quickly complicated, but the procedure is just as straightforward as with single variable calculus.

3 Integrating Multivariable Total Differentials

Integration has many different meanings. Here, I regard the integral as being the opposite of the differential, as a sum instead of a difference. A definite integral, then, is the sum of all infinitely small changes from the starting point to the finishing point.

Notice the usage of the word "point." To do a definite integral, one must identify a complete starting and finishing *point*.

Let us take the right-hand side of (2) as our starting point:

$$x\,\mathrm{d}y + y\,\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}x \tag{12}$$

Now, let's say that we want to find a definite integral for the this. In order to do this, we would need to specify the starting and ending point. Rather than a single value, we would have to specify both an x and a y starting coordinate, and an x and a y ending coordinate.

$$\int_{x=1,y=2}^{x=3,y=-2} (x\,\mathrm{d}y + y\,\mathrm{d}x + 2y\,\mathrm{d}y - \mathrm{d}x) \tag{13}$$

To perform the indefinite part of the integration, we can separate it as follows and perform the integral in a simplified manner:

$$\left(\int (x\,\mathrm{d}y + y\,\mathrm{d}x) + \int (2y\,\mathrm{d}y) - \int (\mathrm{d}x)\right)\Bigg|_{x=1,y=2}^{x=3,y=-2} \tag{14}$$

This can be straightforwardly integrated to our original equation:

$$xy + y^2 - x + C\Big|_{x=1,y=2}^{x=3,y=-2} \tag{15}$$

Evaluating this yields:

$$-3 \cdot 2 + (-2)^2 - 3 - \left(1 \cdot 2 + 2^2 - 1\right) = -9 \tag{16}$$

The meaning of this is that if you sum up all of the changes from one point to the other, then the total of those changes is −9. This result is path-independent—it is just about what the *total* of the changes are. This is obvious, because z is the value of the formula $xy + y^2 - x$, and the total change from one point to the next is literally given by the change in z value, which has no dependence on the path taken.

4 Implicit Functions with Total Differentials

Implicit functions also work with total differentials just the same way. Let us start with the equation

$$\sin(x + z^2) = zy \tag{17}$$

The differential of this can be determined just as before:

$$\mathrm{d}(\sin(x + z^2)) = \mathrm{d}(zy) \tag{18}$$

$$\cos(x + z^2)(\mathrm{d}x + 2z\,\mathrm{d}z) = z\,\mathrm{d}y + y\,\mathrm{d}z \tag{19}$$

$$\cos(x + z^2)\,\mathrm{d}x + 2z\,\cos(x + z^2)\,\mathrm{d}z = z\,\mathrm{d}y + y\,\mathrm{d}z \tag{20}$$

This can be solved for any derivative one wishes. Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\cos(x + z^2)\,\mathrm{d}x + 2z\,\cos(x + z^2)\,\mathrm{d}z = y\,\mathrm{d}z = z\,\mathrm{d}y \tag{21}$$

$$\frac{\cos(x + z^2)}{z} + 2\,\cos(x + z^2)\,\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{22}$$

This, then, has the same basic interpretations as given in Section 1.

References

[1] J. L. Bartlett, "Exploring alternate notations for partial differentials," *Communications of the Blyth Institute*, vol. 1, no. 2, pp. 77–78, 2019.

[2] J. L. Bartlett and A. Z. Khurshudyan, "Extending the algebraic manipulability of differentials," *Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis*, vol. 26, no. 3, pp. 217–230, 2019.