Proving the Derivative of sin(x) Using the Pythagorean Theorem and the Unit Circle

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The derivative of $\sin(x)$ (where x is measured in radians) is given in standard calculus as $\cos(x)$. The proof for this is usually based on a limit: $\lim\limits_{q \to 0} \frac{\sin(q)}{q} = 1$. The proof, put simply, is:

$$y = \sin(x) \tag{1}$$
$$y + \mathrm{d}y = \sin(x + \mathrm{d}x) \tag{2}$$
$$\mathrm{d}y = \sin(x + \mathrm{d}x) - \sin(x) \tag{3}$$
$$\mathrm{d}y = \sin(x)\cos(\mathrm{d}x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{4}$$
$$\mathrm{d}y = \sin(x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{5}$$
$$\mathrm{d}y = \cos(x)\sin(\mathrm{d}x) \tag{6}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)\frac{\sin(\mathrm{d}x)}{\mathrm{d}x} \tag{7}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x) \tag{8}$$

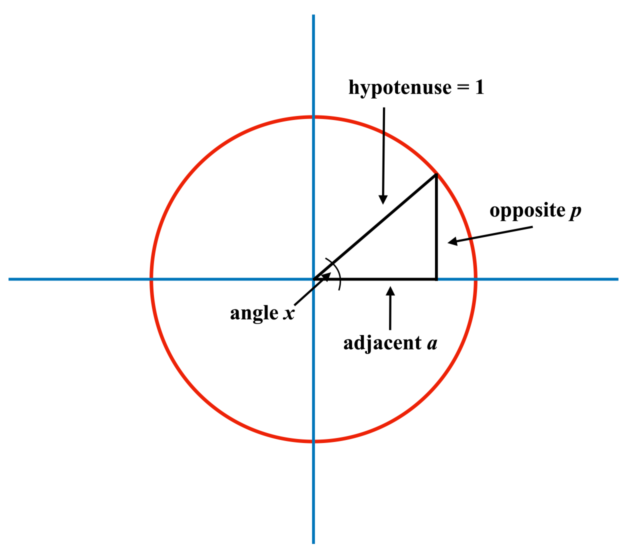
While there is nothing wrong with the proof per se, I have always found it unsatisfying, and, with a dependence on knowing the limit (which is usually found using calculus), a little circular. Therefore, I have endeavored to provide a better proof based on more basic mathematical assertions, founded on the Pythagorean theorem and the unit circle.

I don't know if this proof is unique to me, but I have not found it referenced elsewhere. Therefore, my goal is to make it more widely known, as I think it is both interesting and instructive, as it shows (a) the power of calculus, (b) the power of differential thinking, and (c) how we can make discoveries from basic principles. If it is a new proof, then so much the better.

1 Basic Assumptions

We will be analyzing triangles drawn on the unit circle. On a unit circle, the hypotenuse will always be 1. Figure 1 shows the general setup. x will be the angle measured in radians, a will be the adjacent, and p will be the opposite.

Figure 1: A Triangle Inscribed Onto a Unit Circle



Based on the Pythagorean theorem, we can say the following:

$$a^2 + p^2 = 1 \tag{9}$$
$$p^2 = 1 - a^2 \tag{10}$$
$$a^2 = 1 - p^2 \tag{11}$$
$$\tag{12}$$

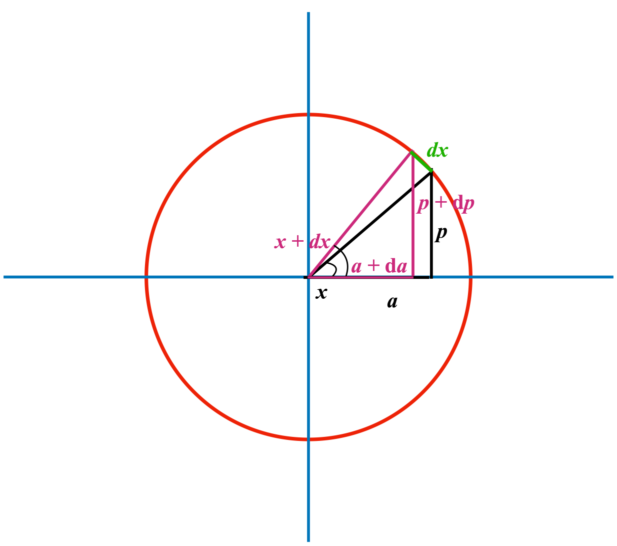
Since the hypotenuse is 1, $\sin(x) = p$ and $\cos(x) = a$. The derivative of $\sin(x)$ with respect to x, therefore, will be $\frac{\mathrm{d}p}{\mathrm{d}x}$. We will be successful if we can prove the following equivalency:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = a \tag{13}$$

2 Differential Analysis

If we take Figure 1, and budge the angle by $\mathrm{d}x$, we will get the picture shown in Figure 2.

Figure 2: Change in Triangle Based on $\mathrm{d}x$



A few important notes on Figure 2:

1. We are expressing all changes as *adding* differentials, even if the differential itself is negative. This is why $a + \mathrm{d}a$ in the graph is shorter than a on its own.
2. Since this is the unit circle, the angle change *is identical* to the circumference change.
3. Since we are only changing by an infinitesimal, and this is a smooth and continuous figure, then the change on the differential is *linear*. In other words, our picture is zoomed in enough that we can treat the arc joining our two triangles as if it were a straight line.

Because of this last point, we can see that length of $\mathrm{d}x$ can be determined using the distance formula, where the horizontal and vertical changes are simply given by $\mathrm{d}a$ and $\mathrm{d}p$:

$$\mathrm{d}x = \sqrt{\mathrm{d}p^2 + \mathrm{d}a^2} \tag{14}$$

Finally, we can take the differential of (9) to come up with:

$$a^2 + p^2 = 1 \tag{15}$$
$$2a\,\mathrm{d}a + 2p\,\mathrm{d}p = 0 \tag{16}$$
$$a\,\mathrm{d}a + p\,\mathrm{d}p = 0 \tag{17}$$
$$a\,\mathrm{d}a = -p\,\mathrm{d}p \tag{18}$$
$$\mathrm{d}a = -\frac{p}{a}\,\mathrm{d}p \tag{19}$$

3 Making the Proof

Starting with (14), we can make substitutions and simplifications as follows:

$$\mathrm{d}x = \sqrt{\mathrm{d}p^2 + \mathrm{d}a^2} \tag{20}$$

$$= \sqrt{\mathrm{d}p^2 + \left(-\frac{p}{a}\,\mathrm{d}p\right)^2} \tag{21}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{p^2}{a^2}\,\mathrm{d}p^2} \tag{22}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{1-a^2}{a^2}\,\mathrm{d}p^2} \tag{23}$$

$$= \sqrt{\mathrm{d}p^2 + \frac{\mathrm{d}p^2}{a^2} - \mathrm{d}p^2} \tag{24}$$

$$= \sqrt{\frac{\mathrm{d}p^2}{a^2}} \tag{25}$$

$$\mathrm{d}x = \frac{\mathrm{d}p}{a} \tag{26}$$

Note that (26) could also have been negative. Inspection of Figure 2 shows that $\mathrm{d}p$ will always have the same sign as a (increasing until a is zero, then decreasing while a is negative). Therefore, choosing the positive square root is the valid choice.

We are trying to figure out an alternative reading of $\frac{\mathrm{d}p}{\mathrm{d}x}$. Using (26), we can simplify this as follows:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\frac{\mathrm{d}p}{a}} = \frac{\mathrm{d}p}{1}\,\frac{a}{\mathrm{d}p} = a \tag{27}$$

As shown in (13), this proves that the derivative of $\sin(x)$ is indeed $\cos(x)$. Additionally, this relies entirely on the basics—the Pythagorean theorem, the unit circle, the definition of sine and cosine, the definition of the radian measure of an angle, the distance formula, and the power rule.