Deciding a Bitstring of 1s is Non-Random is Impossible in General

Eric Holloway

DOI: 10.33014/issn.2640-5652.3.1.holloway.1

Theorem 1 (Chaitin's incompleteness theorem)**.** For every axiomatic proof system T that can be encoded in at least N_T bits, there is a constant c such that the Kolmogorov complexity of a bitstring b cannot be proven to be larger than c.

We can phrase this in terms of an algorithm $A_T(b,n,t)$ that checks every possible proof in T. When applied to a bitstring b and a value n, $A_T(b,n,t)$ outputs a proof in T that $K(b) > n$, or fails after t steps and outputs a null, $\{\}$,

$$\exists c : \forall b, n \geq c, t, A_T(b,n,t) = \{\}.$$

Proof. This is apparent by assuming the opposite,

$$\forall n, \exists b, t, A_T(b,n,t) \neq \{\}. \tag{1}$$

We construct an algorithm B^n_T which, for a given n, performs a breadth first search through all bitstrings b and step amounts t with $A_T(b,n,t)$ until it finds a proof that $K(b) > n$. B^n_T then outputs b. By assumption in Equation 1, B^n_T will always halt.

The Kolmogorov complexity of B^n_T is $K(B^n_T) \leq K(A_T) + \log n + \beta$, where β is a constant for the overhead in B^n_T. The value of β is independent of n, so will not vary as n changes.

We then set $n = c$ such that $K(B^c_T) \leq K(A_T) + \log n + \beta < c$. Since B^c_T will halt, it will output a b such that $K(b) > c$. However, $K(B^c_T) < c$, resulting in a contradiction. Thus, for all $n \geq c$ our assumption in Equation 1 is false.

Therefore, Theorem 1 is true. □

Theorem 2 (Can prove non-randomness)**.** We define a function $U_T(b,n,t)$ which outputs proofs in T of the form $K(b) < \ell(b)$, which are proofs of non-randomness. It is parameterized in the same way as $A_T(b,n,t)$.

For all lengths c of bitstrings b, there is a T that can prove at least one bitstring of length $\ell(b) = c$ is non-random,

$$\exists b \forall c, n, t, U_T(b,n,t) \neq \{\} \wedge \ell(b) = c.$$

Proof. We begin by assuming the contrary

$$\exists c' \forall c > c', n, t, U_T(b,n,t) = \{\} \wedge \ell(b) = c. \tag{2}$$

The following axiomatic system T_{1s} is a falsification of Equation 2.

Axioms of T_{1s} are:

1. $K(\{1\}^{20}) < 20$.
2. If $K(b) < \ell(b)$, then $K(b1) < \ell(b1)$.
3. If $K(b) < \ell(b)$, then b is non-random.

For any bitstring of 1s b_{1s} where $\ell(b_{1s}) \geq 20$, T_{1s} can prove the bitstring is non-random. It does this by incrementally building a bitstring of 1s until the input is matched. The axioms of T_{1s} are true and all proofs are by induction, so all proofs are true. Equation 2 is contradicted. □

Theorem 3 (Cannot generally prove non-randomness)**.** There is no axiomatic system that can decide the non-randomness of every non-random bitstring.

Proof. While a dovetailing algorithm can output proofs of non-randomness for every non-random bitstring, there is no decision procedure that can decide whether the dovetailing algorithm will halt. If there were, then this decision procedure can enumerate all random bitstrings, contradicting Theorem 1. □

Even though a human can trivially decide an arbitrarily long bitstring of 1s is not random, Theorem 3 shows is an impossible task for a generalized algorithm. Only a specific algorithm, such as exemplified in Theorem 2, can do so.

This conclusion is a bit counter-intuitive, since it means that without domain knowledge, an algorithm given an extremely long sequence of 1s would be unsure whether the sequence is completely random. When asked to predict the next digit, the algorithm can only give an equal weighting to 0 and 1.

Proving the Derivative of $\sin(x)$ Using the Pythagorean Theorem and the Unit Circle

Jonathan Bartlett

DOI: 10.33014/issn.2640-5652.3.1.bartlett.1

The derivative of $\sin(x)$ (where x is measured in radians) is given in standard calculus as $\cos(x)$. The proof for this is usually based on a limit: $\lim_{q\to 0} \frac{\sin(q)}{q} = 1$. The proof, put simply, is:

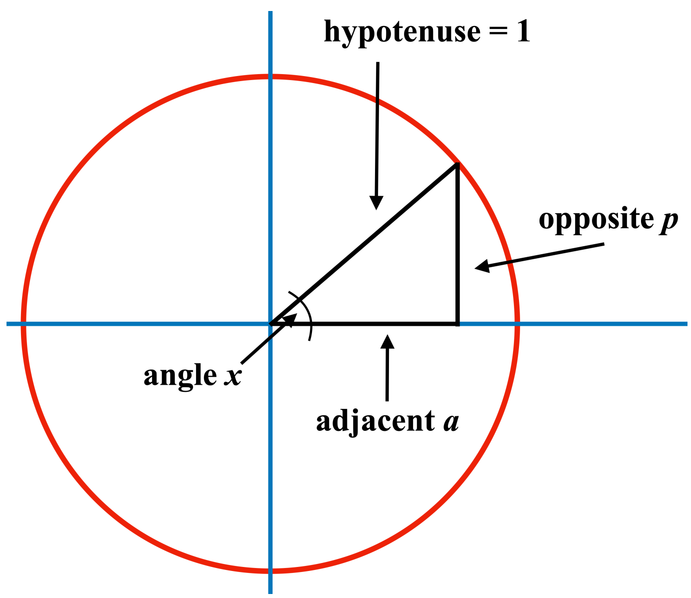
$$y = \sin(x) \tag{1}$$
$$y + \mathrm{d}y = \sin(x + \mathrm{d}x) \tag{2}$$
$$\mathrm{d}y = \sin(x + \mathrm{d}x) - \sin(x) \tag{3}$$
$$\mathrm{d}y = \sin(x)\cos(\mathrm{d}x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{4}$$
$$\mathrm{d}y = \sin(x) + \cos(x)\sin(\mathrm{d}x) - \sin(x) \tag{5}$$
$$\mathrm{d}y = \cos(x)\sin(\mathrm{d}x) \tag{6}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)\frac{\sin(\mathrm{d}x)}{\mathrm{d}x} \tag{7}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x) \tag{8}$$

While there is nothing wrong with the proof per se, I have always found it unsatisfying, utilizing trigonometry identities few students remember. Additionally, it is usually accompanied with an explanation of the limit of $\frac{\sin x}{x}$ that is hard for students to decipher. Therefore, this paper endeavors to provide a more straightforward proof based on more basic mathematical assertions, founded on the Pythagorean theorem and the unit circle. It doesn't remove the given limit in its entirety, but rather gives more straightforward, calculus-oriented reasoning for doing a similar operation. It is debatable how much different it is *in kind* from the standard proof, but in any case I think it is a more straightforward, interesting, and instructive way of looking at it for students. It shows (a) the power of calculus, (b) the power of differential thinking, and (c) how discoveries can be made from basic principles.

Basic Assumptions

This proof will be analyzing triangles drawn on the unit circle. On a unit circle, the hypotenuse will always be 1. Figure 1 shows the general setup. x will be the angle measured in radians, a will be the adjacent, and p will be the opposite.

Figure 1: A Triangle Inscribed Onto a Unit Circle



The Pythagorean theorem gives the following:

$$a^2 + p^2 = 1 \tag{9}$$
$$p^2 = 1 - a^2 \tag{10}$$
$$a^2 = 1 - p^2 \tag{11}$$
$$\tag{12}$$

Since the hypotenuse is 1, $\sin(x) = p$ and $\cos(x) = a$. The derivative of $\sin(x)$ with respect to x, therefore, will be $\frac{\mathrm{d}p}{\mathrm{d}x}$. Therefore, the proof will be successful if it can demonstrate