Letters and Notes

Divergent Series and Its Assigned Value in a Hyperreal Context

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Abstract

This letter discusses the deep connection between the infinite sum of natural numbers and the value $-\frac{1}{12}$. Aside of more widely known facts, we consider a nontrivial way in which we show the veracity of this connection; more precisely this concerns the BGN method (Bartlett, Gaastra, and Nemati, 2020) applied on the so-called damped oscillated Abel summed variant of the series. Moreover, we have found a generalization of this method which 'correctly' assigns finite values to other divergent series. We conclude with some questions concerning whether and how we can analytically relate our hyperreal terms to frame the method in a more justifiable and applicable context.

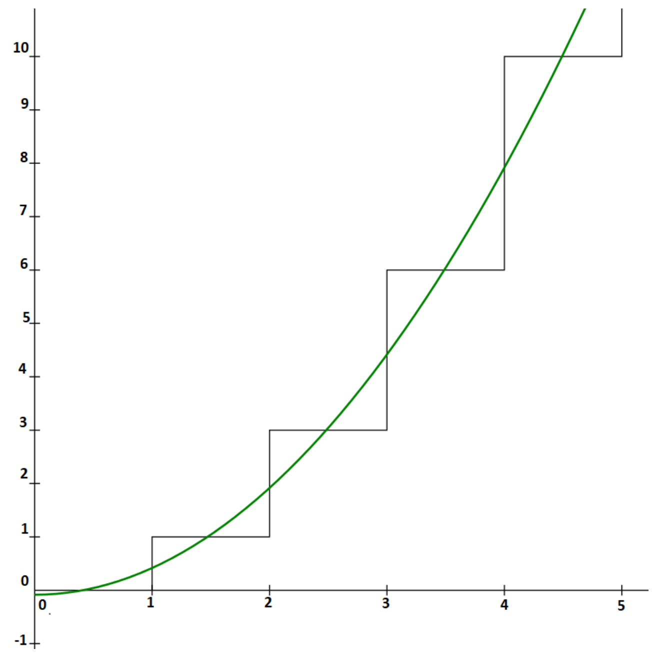

Figure 1: Smoothed partial sums $\sum_{k=1}^{N} k$ with a y-intercept of $-\frac{1}{12}$.

Background

It is obvious that the sum of natural numbers $1+2+3+\cdots$ tends to infinity and can thus not be equal to $-\frac{1}{12}$. There does however exist some connection between this series and value and it is highly probable that this connection is implicitely used (i.e. 'under the hood') in e.g. physics (which often turns out to be perfectly justifiable, as can be shown by various experiments).

The first evidence of this connection is retrieved when one considers the Riemann zeta function $\zeta(s)$. It is known that $\zeta(s)$ is equal to $-\frac{1}{12}$ when $s=-1$ and it is interesting that one retrieves the sum of natural numbers when one 'plugs in' $s=-1$ at the defining series of the Riemann zeta function $\sum_{k=1}^{\infty}\frac{1}{k^s}$. Plugging $s=-1$ in the above series is unfortunately not justifiable (given that $\zeta(s)$ is only equal to this series when $\Re(s)>1$) but it remains an interesting thing to mention.

Another evidence of the connection can be revealed when one considers the 'smoothed version' of the partial sums $\sum_{k=1}^{N} k$, see also Tao, 2010. It turns out that these smoothed partial sums have the same behaviour as the regular partial sums (i.e. they have the same asymptotic expansion) and thus tend to infinity when $N\to\infty$. However, one probably recognizes the constant value in its asymptotic expansion, which is (according to Tao (2010)) given by $CN^2-\frac{1}{12}+O(\frac{1}{N})$ (with C some coefficient of little importance in this case). Moreover, it is given that $-\frac{1}{12}$ is attained when one looks at the intersection with the y-axis in Figure 1.

Damped Oscillations

The first two evidences we mentioned are quite widely known but a more unknown fact can be observed when one considers a damped oscillating variant of the series $1+2+3+\cdots$, namely

$$\sum_{k=1}^{\infty} k e^{-k\epsilon}\cos(k\epsilon). \tag{1}$$

This variant was also discussed in a previous letter (Bartlett and Khurshudyan, 2019). In this letter it was also mentioned that, in the context of hyperreal numbers by introducing $\omega := \infty$ (i.e. by appying the BGN method on it), (1) can be written in a closed-form expression (or at least as an approximation of it). It remained however still unclear which value/magnitude the infinitesemal quantity ϵ must be[1] in order that the BGN method applied on (1) *equals* the 'appropriate' value $-\frac{1}{12}$; only numerical evidence was given. In particular, it was shown that if $\epsilon = \frac{1}{\omega}$, the computing software "Wolfram Mathematica" will include the constant $-\frac{1}{12}$ in its BGN expression (which is similar to the observation of the previous paragraph).

At the time that Bartlett and Khurshudyan (2019) was written, it only seemed clear that there is a numerical evidence that (1) equals $-\frac{1}{12}$ when we take ϵ in a sufficiently small interval. Recently, we have found that Sugiyama (2014) (Section 2.3) provides a more theoretical derivation of this matter. Although the website and its choice of words are somewhat vague and confusing, the derivation seems correct. In this derivation there is being made use of a so-called 'damped oscillated Abel summation method', which is a kind of generalization of the more common Abel summation method used to assign finite values to divergent series. In this article, this method of 'damped oscillated Abel summation' is consequently used on a larger class of divergent series as well; furthermore it turns out that the 'damping' and 'vibrating' constant should not be necessarily equal to each other. We thus in fact have that (see also Section 5.2 and Section 6.1 of Sugiyama (2014), we here write ϵ instead of x) $\sum_{k=1}^{\infty} k^i$ can be transformed to (letting $i \geq 1$ be an integer)

$$\sum_{k=1}^{\omega} k^i e^{-k\epsilon \cot \frac{\pi}{2i+2}} \cos(k\epsilon)$$

and

$$\sum_{k=1}^{\omega} k^i e^{-k^{\frac{i+1}{2}}\epsilon} \cos(k^{\frac{i+1}{2}}\epsilon)$$

and consequently taking the limit $\epsilon \to 0$ yields the 'appropriate' assigned value; we also numerically verified this[2].

It remains of course interesting how this damped oscillated Abel summation method can be stated in our more 'detailed' hyperreal context; i.e. in which we know the exact values of ϵ (possibly in terms of ω) in order that the BGN method assigns the 'appropriate' value to a divegent series. Unless it is still untrivial which values ϵ must have in order that the mentioned method assigns this value, we can however say from Equation (5.57) in Sugiyama (2014) that in general $\frac{1}{\epsilon}$ must be a lot smaller than ω (this was also shown by numerical experiments: if we set $\epsilon = 0.01$, ω must be a lot larger than 100).

[1] in relation to ω

[2] by again letting ϵ be in a sufficiently small interval

Conclusion

In conclusion, we can thus say that the connection between the often assigned value of a divergent series is hidden in its asymptotic expansion. Furthermore, some slight variations (performed in the context of hyperreals) of the terms in the divergent series will alterate its asymptotic expansion in such a manner that that the BGN method assigns the 'desired value' to it. As it is at this point still untrivial when equality holds, and how in this case ϵ and ω thus must be related, remains an interesting topic for further research. To state this in a more general and mathematically way: Consider a divergent series with BGN expansion $A(\epsilon(\omega))\omega^2 + C + O(1/\omega)$ (here A is a value dependent of ϵ which is in turn dependent of ω and C is the 'appropriate' value we want to have), the question is now which variations (in terms of $\epsilon(\omega)$) we have to make in order to make $A(\epsilon(\omega))\omega^2$ equal to zero.

Bartlett, J, L Gaastra, and D Nemati (2020). "Hyperreal Numbers for Infinite Divergent Series". In: *Communications of the Blyth Institute* 2.1, pp. 7–16.

Bartlett, J and A Khurshudyan (2019). "Numberphile's Proof for the Sum 1+2+3+..." In: *Communications of the Blyth Institute* 1.1, pp. 54–55.

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