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Numberphile's Proof for the Sum $1 + 2 + 3 + \ldots$

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DOI: 10.33014/issn.2640-5652.1.1.bartlett.3

In 2014, YouTube math vlogger Numberphile upset the amateur math world by declaring that the sum of all the natural numbers (i.e., the positive integers, the infinite series $1 + 2 + 3 + \ldots$) is $-\frac{1}{12}$ (Haran and Padilla, 2014; Haran, 2015). While this is indeed the result of the Riemann Zeta function applied to -1, we will show here that it is not the sum of $1 + 2 + 3 + \ldots$.

The standard summation which the Riemann Zeta function is based on is simple:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}. \tag{1}$$

For $x > 1$, (1) is well defined and makes a convergent series. For $x \leq 1$, (1) no longer converges. For $x = -1$, (1) is equivalent to the series under consideration, $1 + 2 + 3 + \ldots$. The question is whether or not $\zeta(-1)$ is still equivalent to the series implied by (1). If it is, then $1+2+3+\ldots$ is indeed equal to $-\frac{1}{12}$.

According to the video, which uses a proof based on the one originally given by Ramanujan, the proof that $1+2+3+\ldots = -\frac{1}{12}$ can be shown by beginning as follows. First, start with the following series:

$$S_1 = 1 - 1 + 1 - 1 \ldots \tag{2}$$

$$S_2 = 1 - 2 + 3 - 4 \ldots \tag{3}$$

$$S_3 = 1 + 2 + 3 + 4 \ldots \tag{4}$$

S_1 has the well-known value of $\frac{1}{2}$ and S_2 has the well-known value of $\frac{1}{4}$. He then subtracts $S_3 - S_2$. Doing this yields the series $0 + 4 + 0 + 8 \ldots$.

The error comes next. This is claimed to be equivalent to the series $4 + 8 + 12 \ldots$, which would be $4S_3$. This gives the equation $S_3 - S_2 = 4S_3$. Because $S_2 = \frac{1}{4}$, this can be then solved.

$$S_3 - \frac{1}{4} = 4S_3 \tag{5}$$

$$3S_3 = -\frac{1}{4} \tag{6}$$

$$S_3 = -\frac{1}{12} \tag{7}$$

As suggested the problem comes with stating that

$$0 + 4 + 0 + 8 + 0 + 12 \ldots = 4 + 8 + 12 \ldots.$$

Bartlett, Gaastra, and Nemati (2018) developed a technique, which we can term the BGN technique, that assigns hyperreal values to divergent sums. Applying the BGN technique shows that, even though it may seem counterintuitive, adding zeroes in the middle of an infinite sum changes the value of the sum, therefore invalidating the proof.

According to the method given in Bartlett, Gaastra, and Nemati (2018), the sum of $1 + 2 + 3 \ldots$ is the hyperreal value $\frac{\omega^2}{2} + \frac{\omega}{2}$. The sum of $0+4+0+8+0\ldots$ is the hyperreal value $\frac{\omega^2}{2} + \frac{\omega}{2} - \frac{1}{4}$, which, in fact, is the result of $S_3 - \frac{1}{4}$. Note that both of these series are essentially the same value, as the lower-orders of infinity are essentially noise compared with the highest order term, which is $\frac{\omega^2}{2}$.

The reason why $\zeta(-1) = -\frac{1}{12}$ while the series (1) doesn't is that $\zeta(-1)$ is evaluated using the Zeta function's *analytic continuation* (a modification of a function that expands its domain), *not* the series given in (1). The analytic continuation of ζ (the expanded expression that *actually is* valid for -1) is, according to Lavrik (2011),

$$\pi^{-x/2}\Gamma\left(\frac{x}{2}\right)\zeta(x) = \frac{1}{x(x-1)} + \int_1^\infty \left(x^{-(1-x/2)} + x^{-(1-(1-s)/2)}\right)\theta(x)\,\mathrm{d}x, \quad (8)$$

where Γ is the Euler Gamma Function, and $\theta(x)$ is $\sum_{n=1}^\infty e^{-\pi n^2 x}$. This is no longer identical to the original expression given in (1).

However, the question still remains why physicists can use $-\frac{1}{12}$ as a stand-in for the sum of all natural numbers. As Haran and Padilla (2014) point out, in several aspects of physics, such as for the Casimir effect, when physicists need a sum of all natural numbers, the Zeta function can act as a stand-in and yield valid results.

While no conclusive reason for this has been established, Vandegrift (2014) offers a numerical evaluation of a series that is very similar to the series $1+2+3\ldots$, but is offset by a tiny complex component. Vandegrift has suggested the possibility that

$$\lim_{\epsilon\to 0}\sum_{n=1}^\infty n e^{-\epsilon n}\cos(\epsilon n) = -\frac{1}{12}. \quad (9)$$

This sum would be nearly identical to $1 + 2 + 3 \ldots$ in its beginning, but begin to diverge for higher values of n. We investigated this possibility and found the following results:

1. For an infinitesimal ϵ (where $\epsilon = \omega^{-1}$), the series actually diverges.
2. Interestingly, in the evaluation of the series expansion of (9), even though it diverges to infinity, there is a component of it that is $-\frac{1}{12}$.[2]
3. For a finite ϵ, a wide range of values will produce results near $-\frac{1}{12}$, though we did not yet find a value that produces this value exactly. ϵ ranging from $\frac{1}{2}$ to $\frac{1}{3750}$ seemed to be fairly close, while values outside this range started to stray.

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[2] Using the BGN technique, the expansion of the sum was found to be

$$\left(\frac{\sin(1)}{e} - \frac{\cos(1)}{2e}\right)\omega^2 + \left(\frac{\cos(1)}{2e}\right)\omega - \frac{\sin(1)}{12e} - \frac{1}{12}$$

where ω is the hyperreal infinite unit. Notice the last part of this term is $-\frac{1}{12}$. It is unclear the connection between this value and the Zeta function. Nonetheless, it is interesting that $-\frac{1}{12}$ appears there.